Matrix and Imaging

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OUTLINE

Matrices and Images

Image Restoration

THE MATRIX

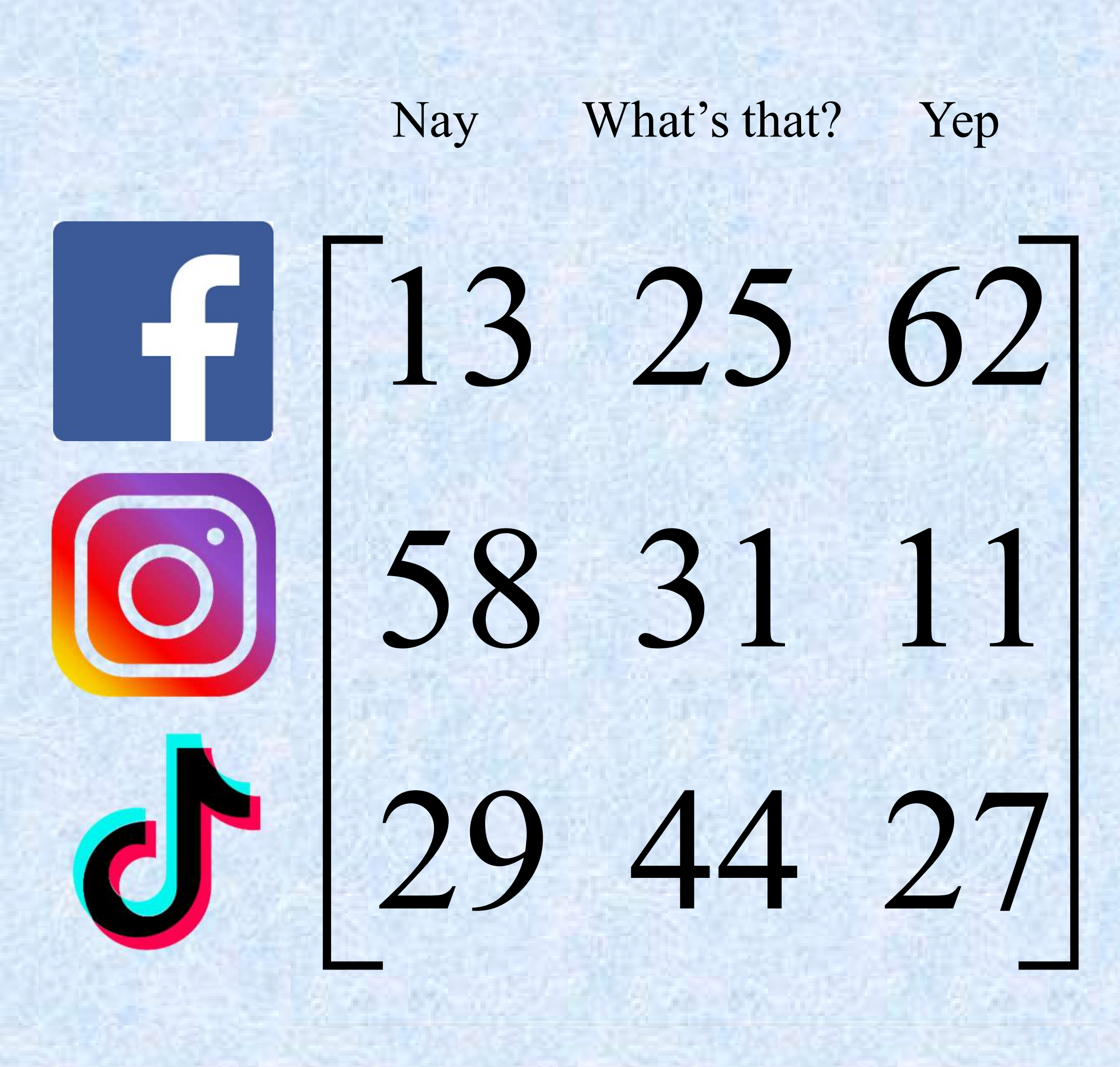
https://commons.wikimedia.org/wiki/File:Digital_rain_animation_medium_letters_shine.gif
https://screenrant.com/matrix-4-original-movies-look-different-comparison-upgrades/

You know what I'm talking about?

THE MATRIX (1999)

Matrix is Everywhere

What social platform you use?



$$\begin{array}{c} \text{Nay} \quad \text{What's that?} \quad \text{Yep} \\ \begin{bmatrix} 13 & 25 & 62 \\ 58 & 31 & 11 \\ 29 & 44 & 27 \end{bmatrix} \end{array}$$

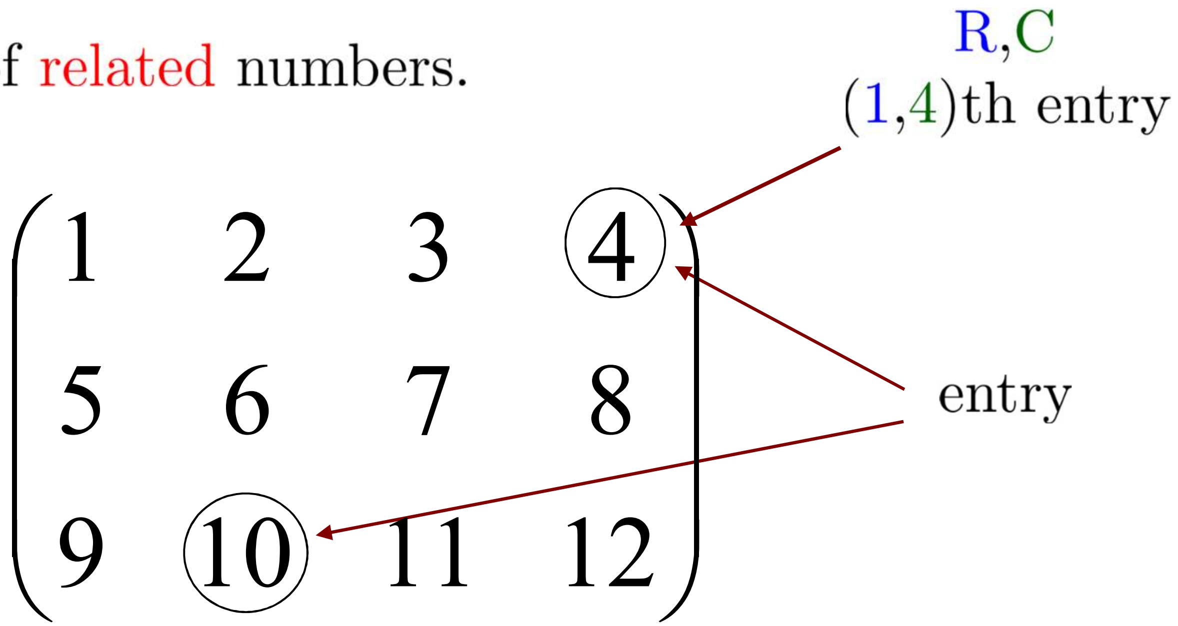
matrix = table

but with

$+, -, \times, \div$

What is a Matrix

A matrix is a table of related numbers.



R,C
(1,4)th entry

$$\begin{pmatrix} 1 & 2 & 3 & \textcircled{4} \\ 5 & 6 & 7 & 8 \\ 9 & \textcircled{10} & 11 & 12 \end{pmatrix}$$

entry

A 3×4 matrix (3 rows and 4 columns)

R, C = Raymond Chan

Large number of related data ⟹ matrices ⟺ big data

Matrix Addition and Subtraction

$$\begin{pmatrix} \enclose{circle}{1} & 2 & 3 \\ 4 & 5 & \enclose{circle}{6} \end{pmatrix} + \begin{pmatrix} \enclose{circle}{7} & 8 & 9 \\ 10 & 11 & \enclose{circle}{12} \end{pmatrix} = \begin{pmatrix} \enclose{circle}{1+7} & 2+8 & 3+9 \\ 4+10 & 5+11 & \enclose{circle}{6+12} \end{pmatrix}$$

entry-wise addition

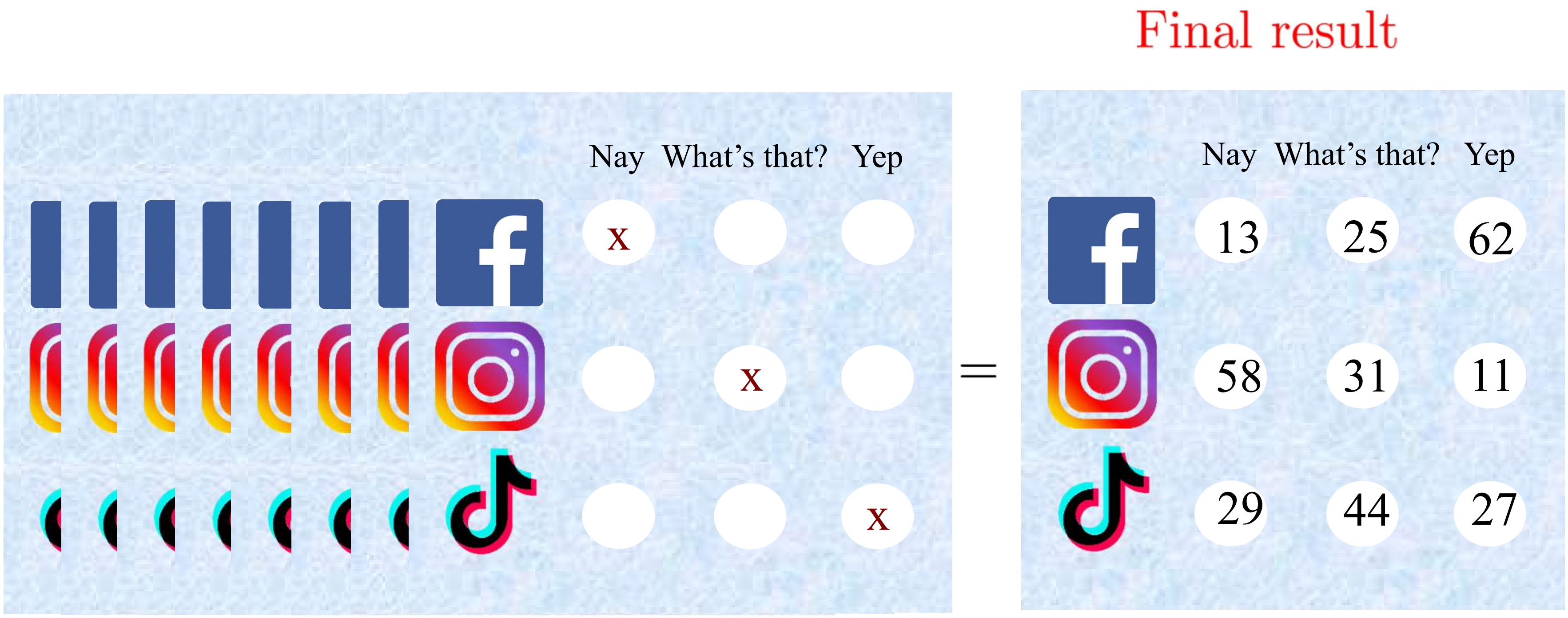
$$= \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 & \enclose{circle}{3} \\ 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 7 & 8 & \enclose{circle}{2} \\ 2 & 1 & 12 \end{pmatrix} = \begin{pmatrix} 5-7 & 2-8 & \enclose{circle}{3-2} \\ 4-2 & 5-1 & 6-12 \end{pmatrix}$$

entry-wise subtraction

$$= \begin{pmatrix} -2 & -6 & 1 \\ 2 & 4 & -6 \end{pmatrix}$$

Why Entry-wise Addition and Subtraction?



$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \cdots + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 25 & 62 \\ 58 & 31 & 11 \\ 29 & 44 & 27 \end{pmatrix}$$

Matrix Multiplication

$$\begin{array}{cc} & \begin{array}{cc} & \begin{array}{cc} C & C \end{array} \end{array} \\ \begin{array}{c} R \\ R \end{array} & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \end{array} = \begin{pmatrix} \overset{R}{\begin{pmatrix} 1 & 2 \end{pmatrix}} \overset{C}{\begin{pmatrix} 5 \\ 7 \end{pmatrix}} & \overset{R}{\begin{pmatrix} 1 & 2 \end{pmatrix}} \overset{C}{\begin{pmatrix} 6 \\ 8 \end{pmatrix}} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

inner products

Why not entry-wise?

Inner Products

聚寶華軒
Gourmet Chinese Restaurant
點心記錄咭 年 月 日
檯號 經手人 茶 芥
68259

$8	小點	⑤ ⑤	2
$10	中點	⑧	1
$12	大點	③	1
$15	特點	⑦ ⑦	2
$18	頂點		0

Dim-sum record card

$$8\times 2+10\times 1+12\times 1$$
$$+15\times 2+18\times 0$$

$$= \underset{R}{\begin{pmatrix}8 & 10 & 12 & 15 & 18\end{pmatrix}}\overset{C}{\begin{pmatrix}2\\1\\1\\2\\0\end{pmatrix}}$$

inner products

Matrix Multiplication and Division

Matrix is a mathematical tool to represent the relationship between action and reaction (observation) of physical or economic phenomena.

$$\textbf{matrix} \longrightarrow A\underset{\underset{\textbf{action}}{\nearrow}}{\mathbf{x}} = \underset{\underset{\textbf{reaction}}{\nwarrow}}{\mathbf{b}}$$

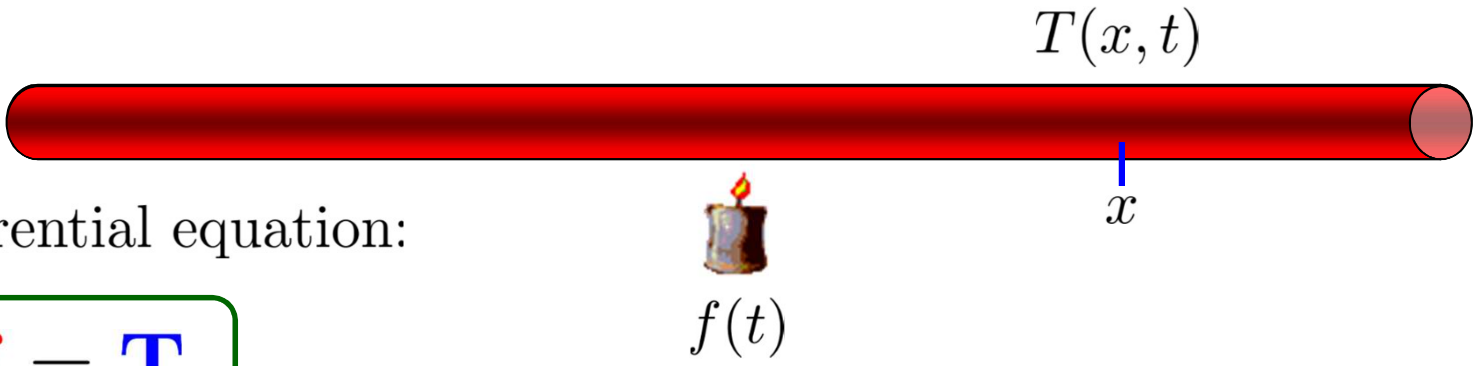
Matrix multiplication enables us to do prediction and forecasting.

To know what action $\mathbf{x}$ causes reaction $\mathbf{b}$, we need matrix division.

$$\textbf{action} \longleftarrow \mathbf{x} = A^{-1}\mathbf{b} \longleftarrow \textbf{reaction}$$

Reverse process enables us to do planning or restoration.

Heat Equation



Partial differential equation:

$$\boxed{A\mathbf{f} = \mathbf{T}}$$

$$\boxed{\mathbf{f} = A^{-1}\mathbf{T}}$$

- ☐ Same equation for option pricing: (1997 Nobel Prize in Economics)

Underlying Code	Underlying Name	HKATS Code	Call / Put	Expiry (D/M/Y)	Strike
00001	CKH Holdings	CKH	Call	28/05/20	52.50
00001	CKH Holdings	CKH	Call	28/05/20	55.00
00001	CKH Holdings	CKH	Call	28/05/20	57.50
00001	CKH Holdings	CKH	Put	28/05/20	52.50
00001	CKH Holdings	CKH	Put	28/05/20	55.00
00001	CKH Holdings	CKH	Put	28/05/20	57.50

- ☐ Same equation for image denoising: (First picture of black holes)

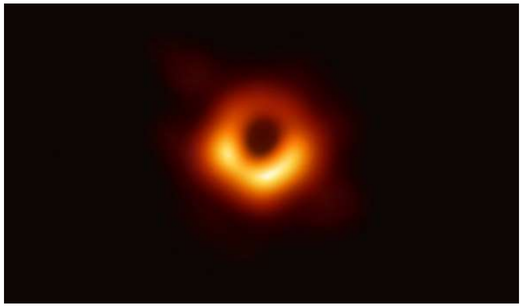


Image credit: https://www.sciencenewsforstudents.org/

What is an Image?

Grayscale image of the moon as seen by human.

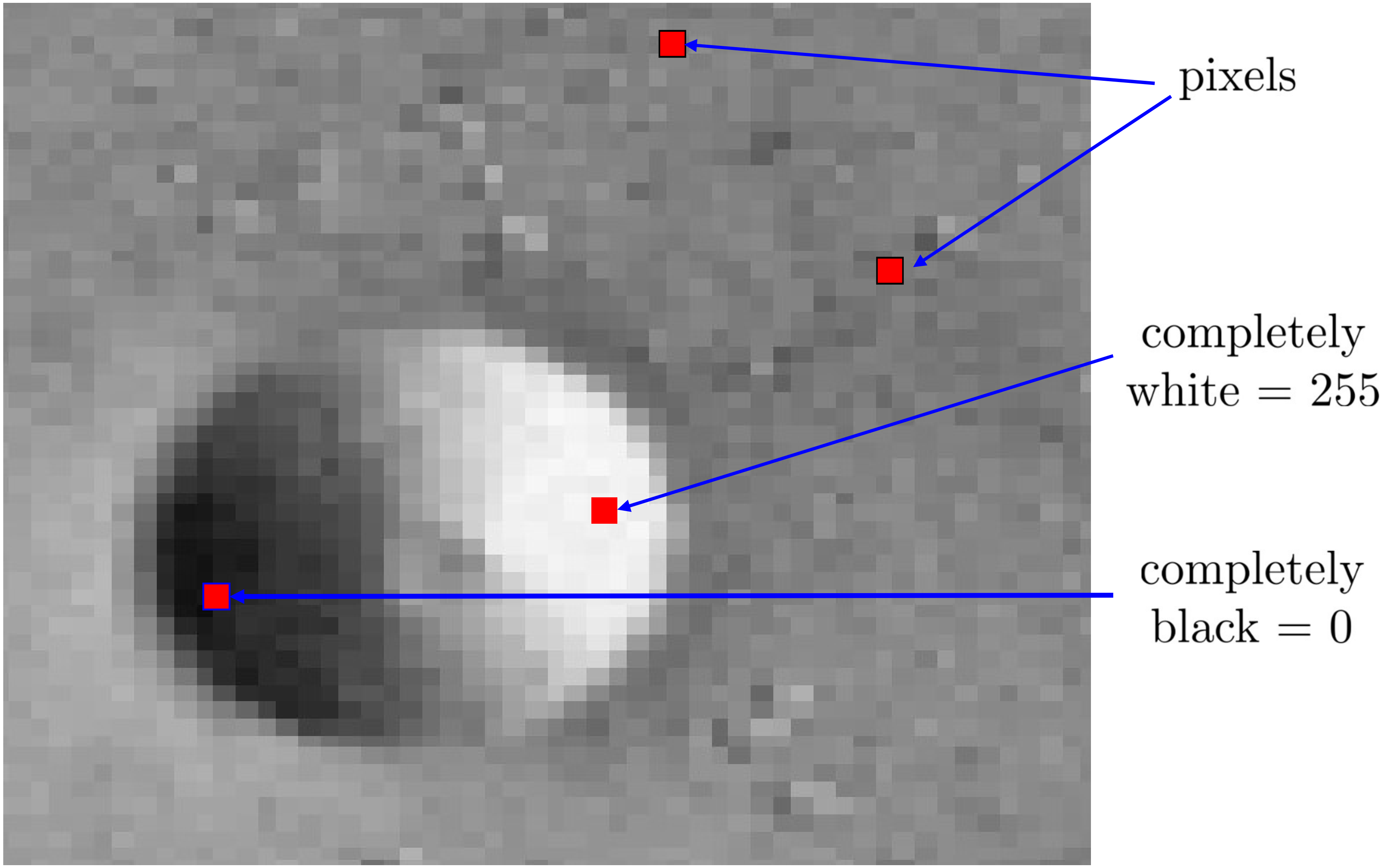
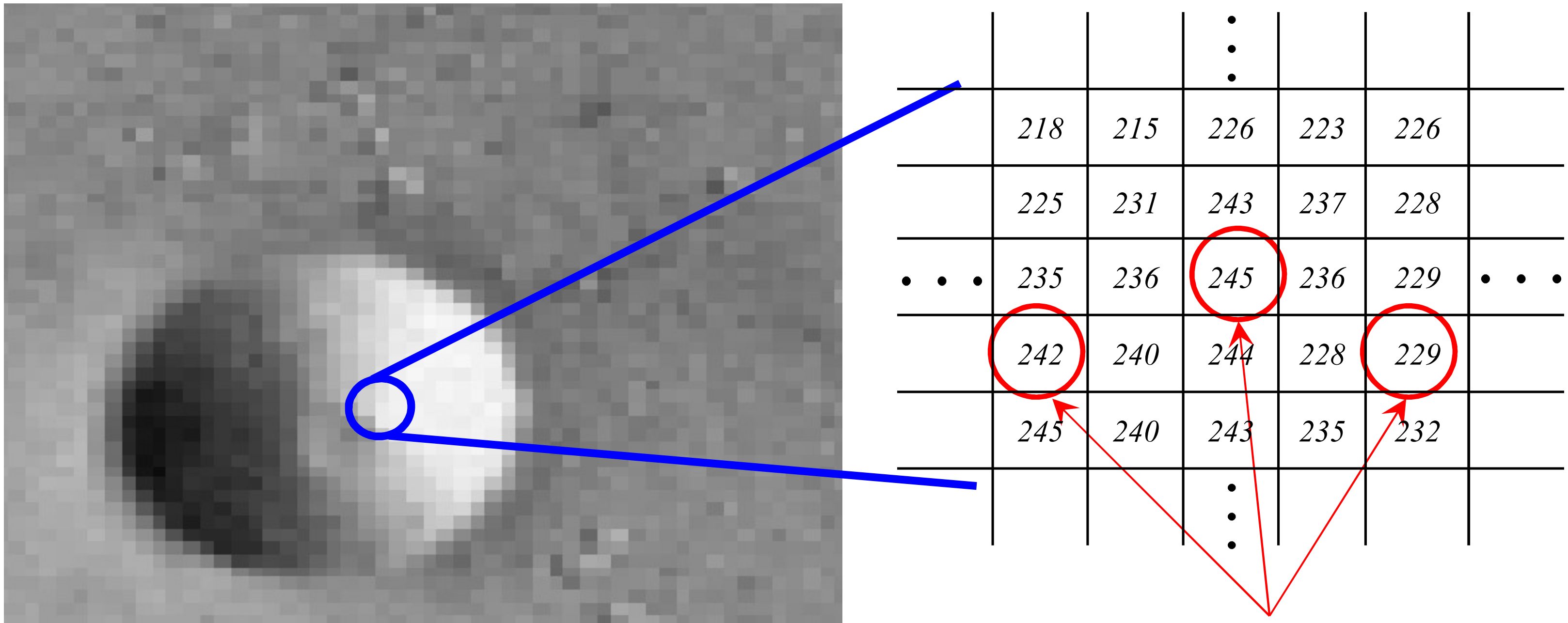


Image as Matrix

Grayscale image of the moon as seen by a computer.



218	215	226	223	226
225	231	243	237	228
235	236	245	236	229
242	240	244	228	229
245	240	243	235	232

pixel values
between 0 and 255

$$\text{pixel value between 0 and 255} = 8 \text{ bits/pixel}$$

4000-by-4000 image = 4000-by-4000 matrix = 16 Mbyte

Color Images

RGB (red, green and blue channels) 24-bit color:

color image

red channel

4000-by-4000 color image = three 4000-by-4000 matrices

= 48 MBytes memory

OUTLINE

Matrices and Images

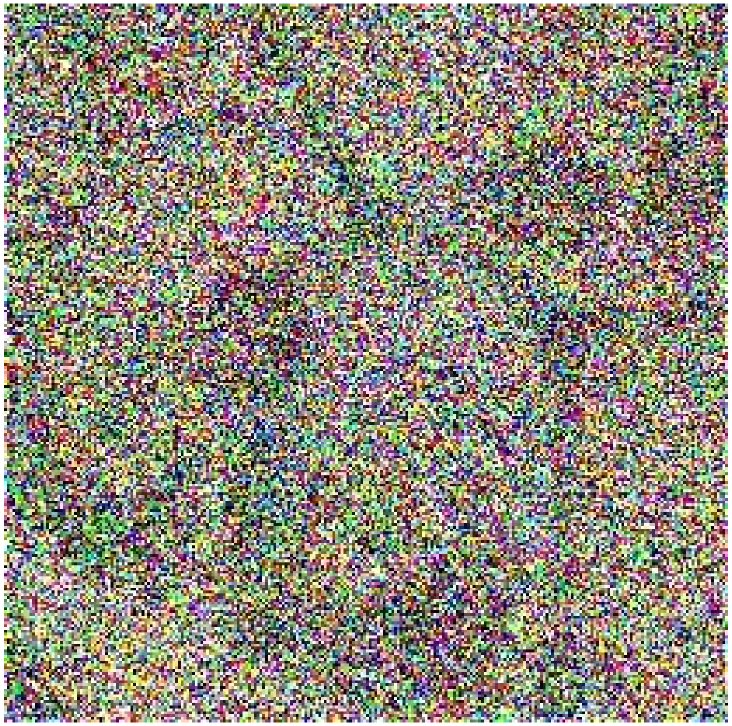
Image Restoration

Image Restoration

From dark to bright

From blur to sharp

Removing Random Noise

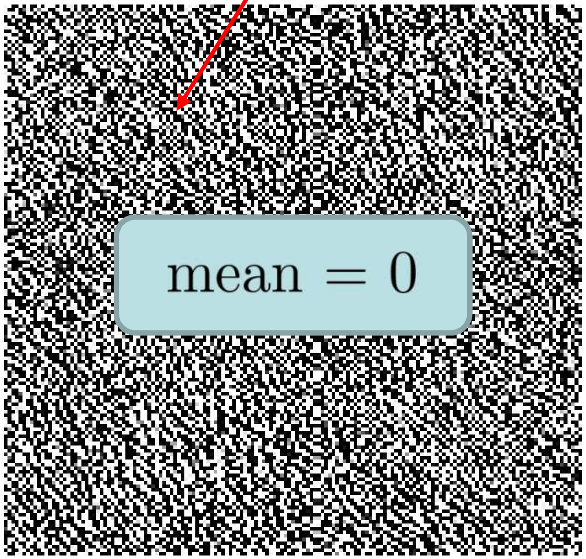


Cells under a microscope corrupted by random noise

true image

$$= I + N$$

noise



Averaging Out the Noise

Since the noises are random with mean 0, we take k images and average them:

$$\frac{(I + N_1) + (I + N_2) + \ldots + (I + N_k)}{k}$$

$$= \quad I + \frac{N_1 + N_2 + \ldots + N_k}{k} \approx I$$

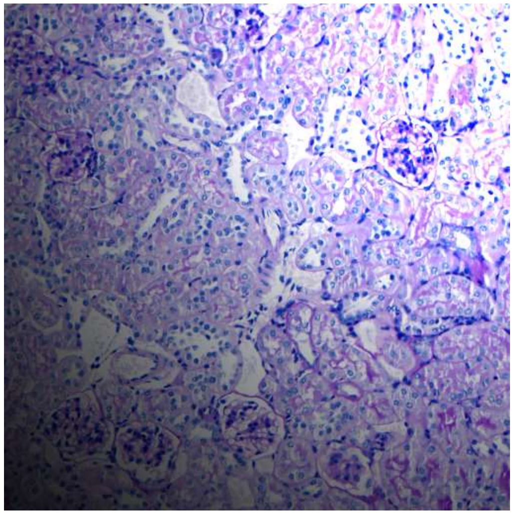
Average of 1024 images

The noise will kill each other and we have an image with very little noise.

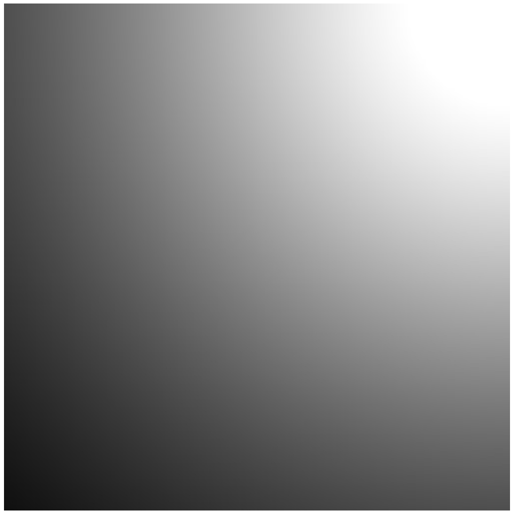
Matrix addition

Removing Background Noise

A = Image with light from NE

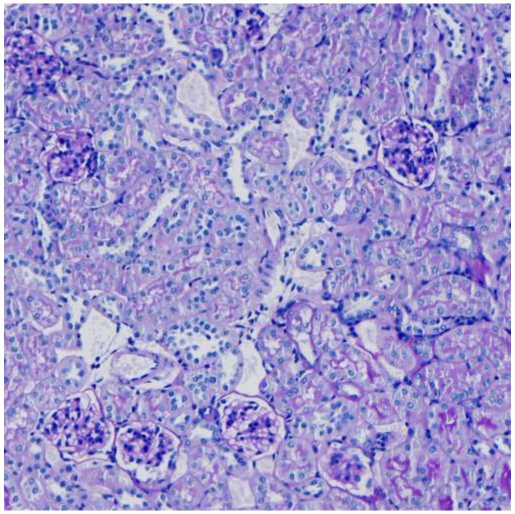


B = background light source



Matrix subtraction

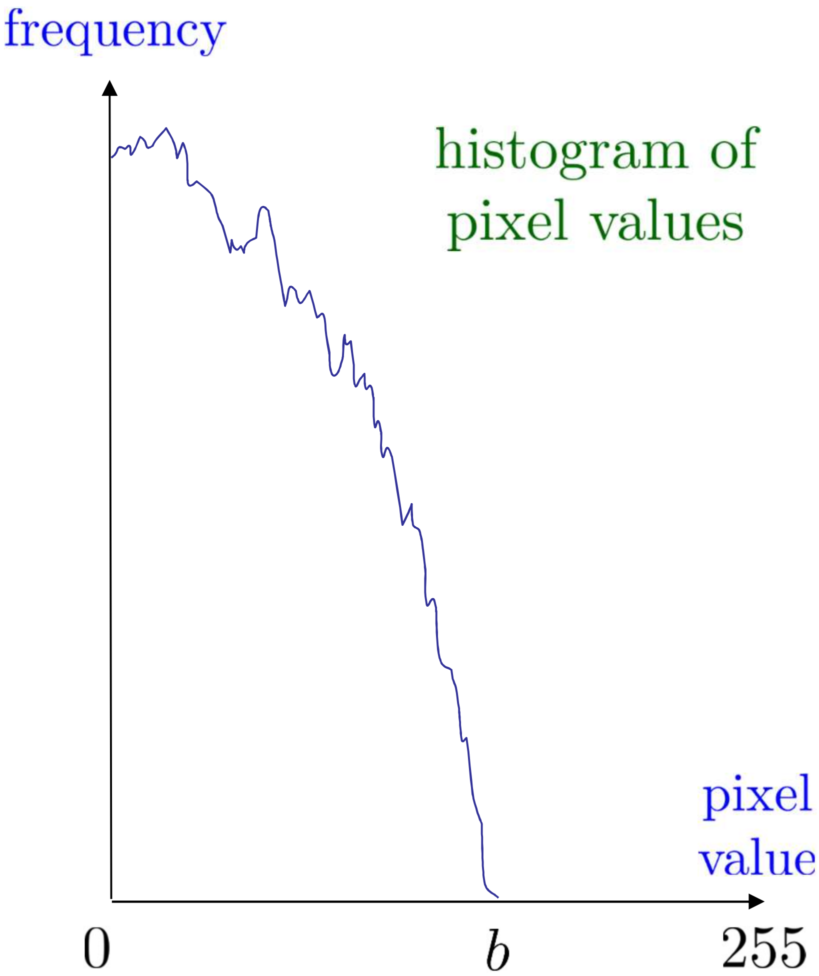
$$I = A - B$$



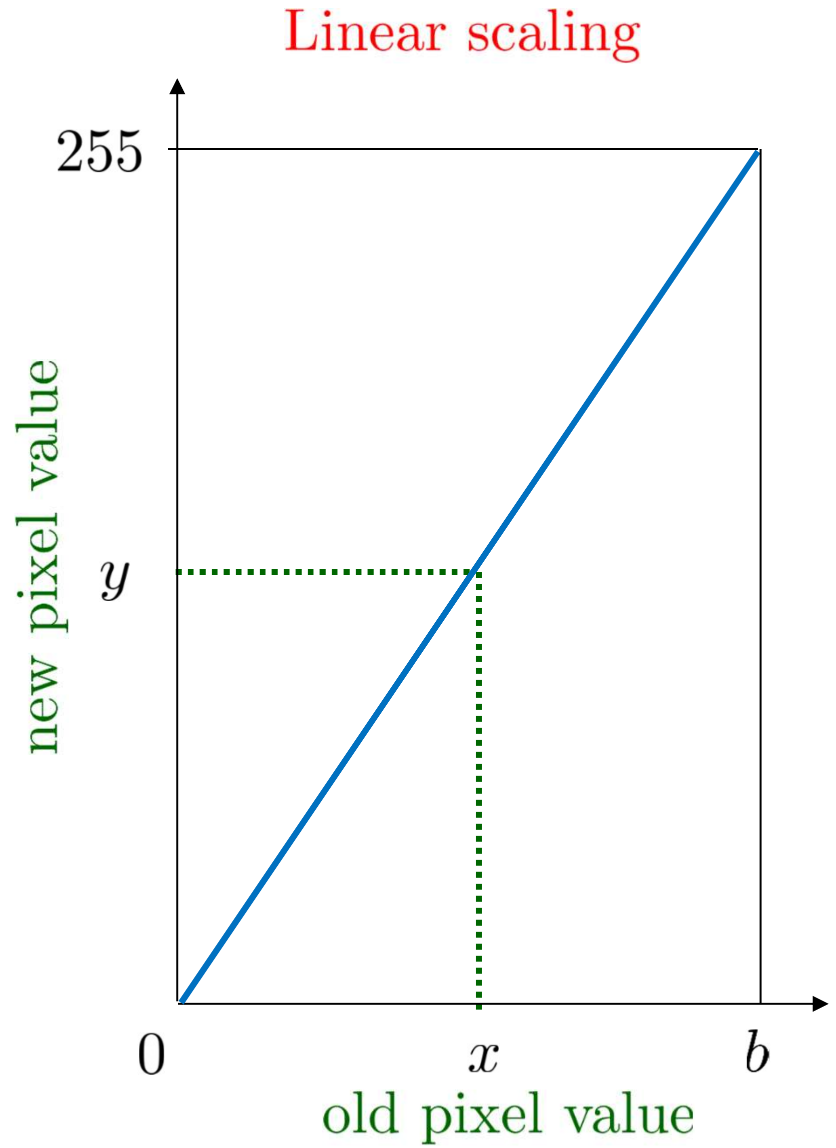
Contrast Enhancement

Picture with low contrast

Pixel values concentrate in $(0, b)$, where $b \ll 255$.



Linear Scaling to Increase Contrast


Linear scaling

Scalar Multiplication

$$y = \frac{255}{b} \cdot x$$

$$Y = \frac{255}{b} \cdot X$$

Deblurring

image due to
motion to
the right

Motion blur

The Blurring Process

Observed image $\mathbf{p}$ is obtained from true image $\mathbf{q}$ as follows:

$= \mathbf{p} =$

$$\mathbf{p}(i) = \mathbf{q}(i) + \mathbf{q}(i+1) + \mathbf{q}(i+2) + \cdots + \mathbf{q}(i+k)$$

$\mathbf{p}$ accumulates many translated copies of $\mathbf{q}$ before the shutter closes.

The Blurring Matrix

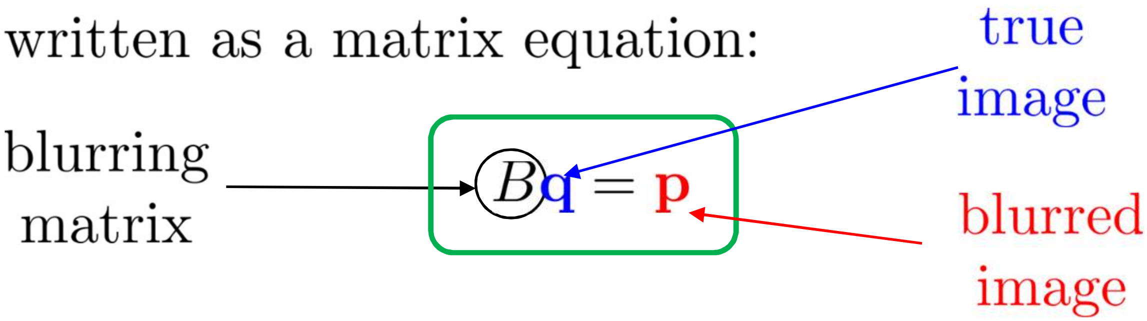
Therefore:

$$\mathbf{p}(i) = \sum_t \mathbf{q}(i+t).$$

More generally, blurring process can be written as:

$$\mathbf{p}(i) = \sum_t B(t)\mathbf{q}(i+t).$$

This can be written as a matrix equation:



blurring matrix

$$\boxed{B\mathbf{q} = \mathbf{p}}$$

true image

blurred image

To obtain the true image $\mathbf{q}$:

$$\boxed{\mathbf{q} = B^{-1}\mathbf{p}}$$

Matrix division

Out-of-Focus Blur

The observed image **p** due to wrong focus:

- ☐ Difficult to invert F?
- ☐ How big is matrix F?
- ☐ How big is your image?
- ☐ For 1000×1000 image F is $1000^2 \times 1000^2$
- ☐ 1 trillion entries in F
- ☐ 10^{18} operations to invert F
- ☐ Need 100 years on a PC

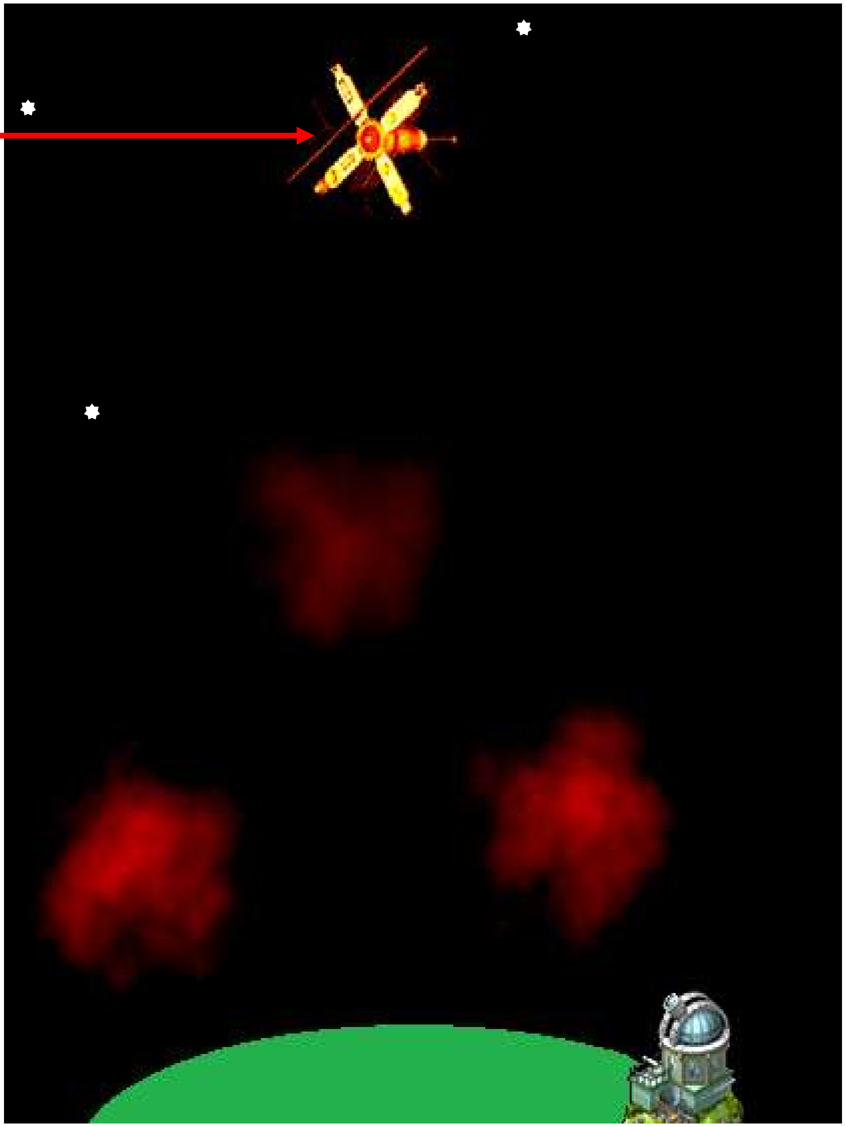
another blurring matrix →

$$\boxed{F\mathbf{q} = \mathbf{p}} \longrightarrow \boxed{\mathbf{q} = F^{-1}\mathbf{p}}$$

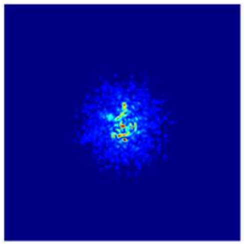
We need mathematics!

Ground-based Astronomy

true image $\mathbf{q}$

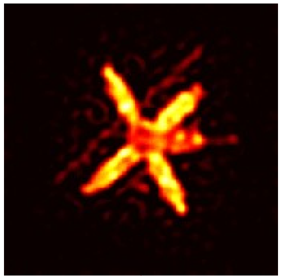


twinkle,
twinkle,
little star

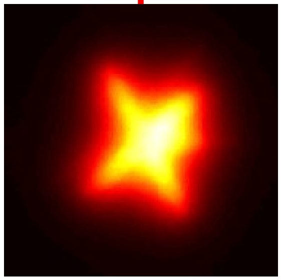


atmospheric
blurring
A

restored image

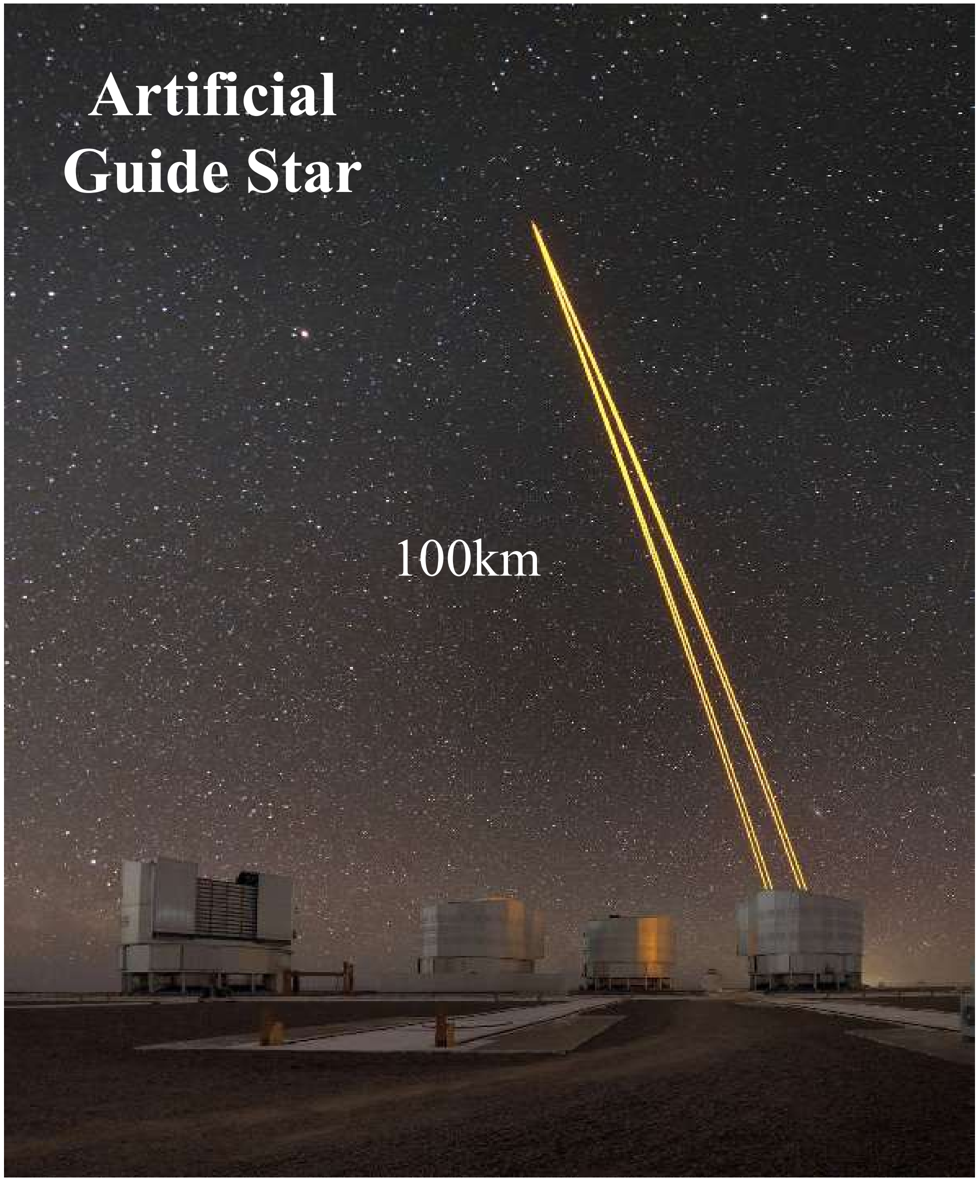


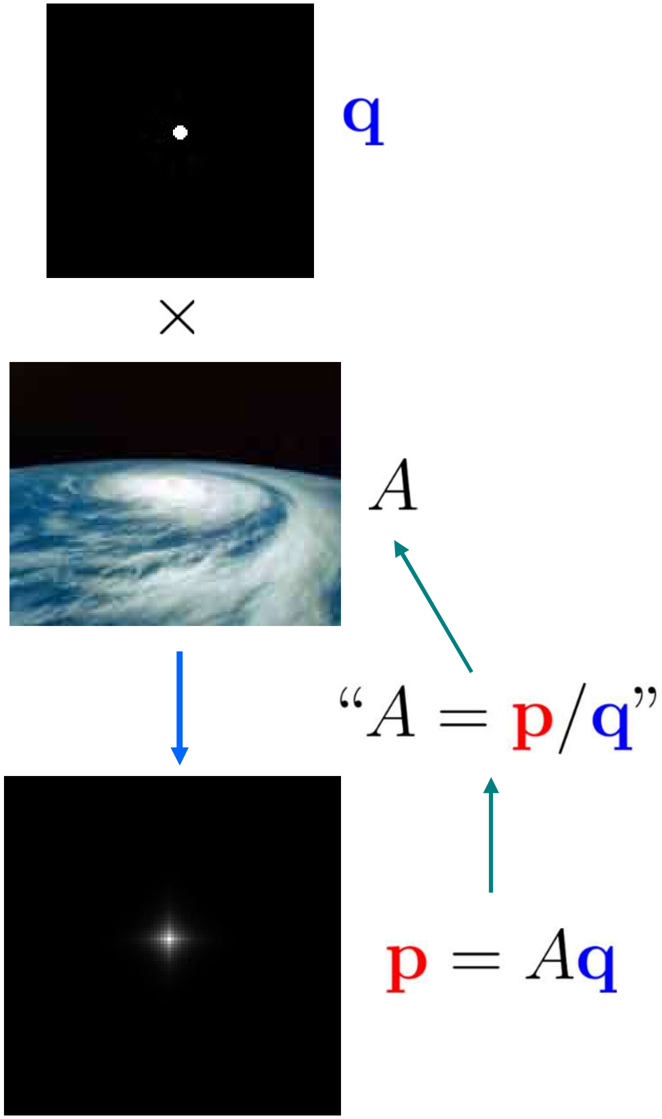
$\mathbf{q} = A^{-1}\mathbf{p}$



observed image
$\mathbf{p} = A\mathbf{q}$

How to Determine Atmospheric Blurring?





$\mathbf{q}$

$\times$

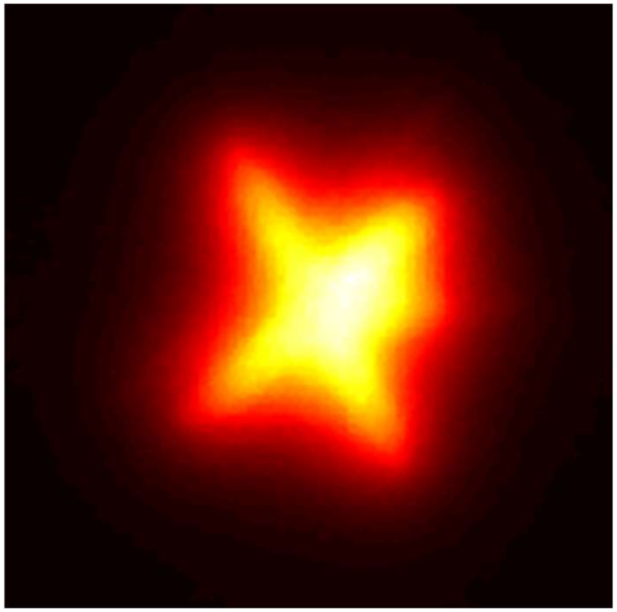
A

“$A = \mathbf{p}/\mathbf{q}$”

$\mathbf{p} = A\mathbf{q}$

Image credits: National Geographic, European Southern Observatory

Restoration of Satellite Image



Satellite from
Ground-based telescope

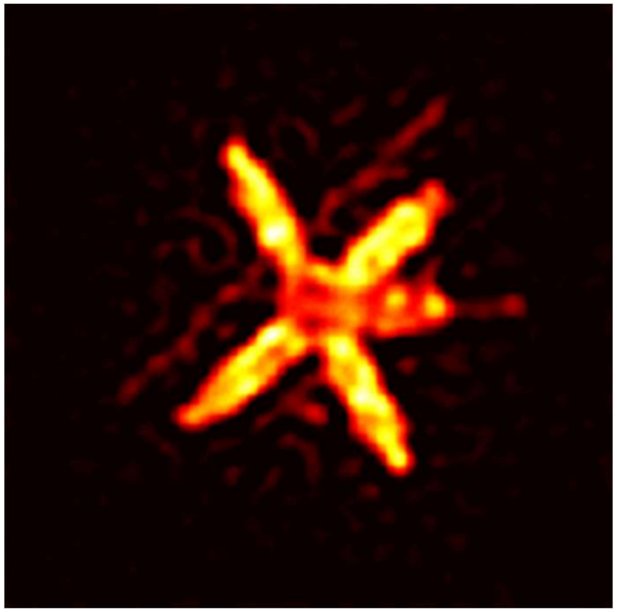
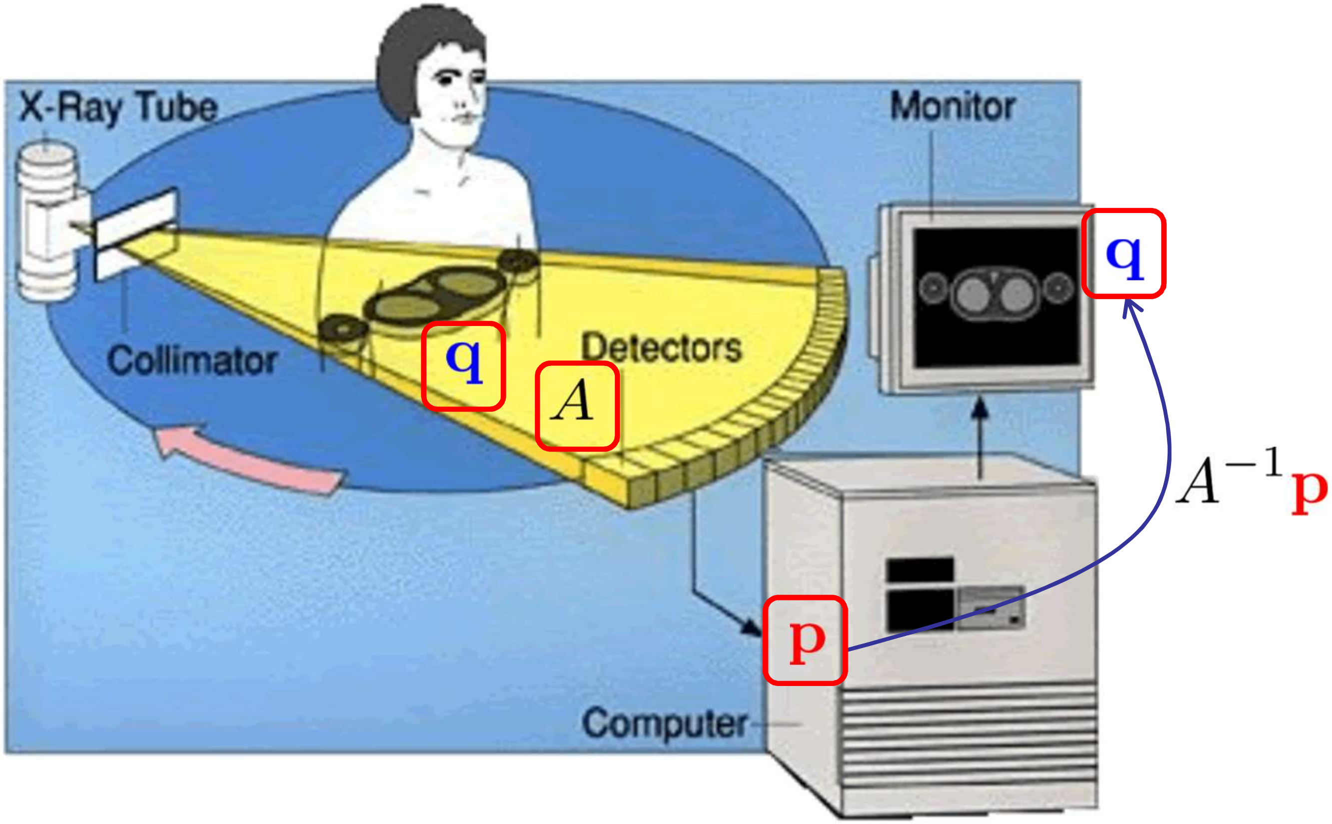


Image after
reconstruction
(1993)

https://faculty.sites.wfu.edu/plemmons/a-clear-view-of-forever
https://faculty.sites.wfu.edu/plemmons/congressional-testimony

Computed Tomography (CT) Scan

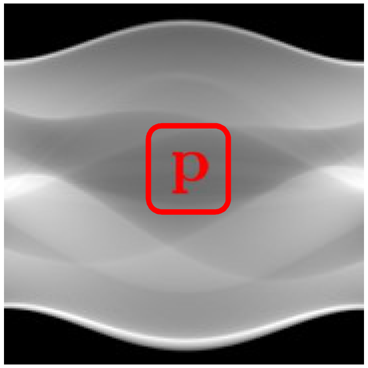


CT system

https://global.medical.canon/

Integral equation:

$$\boxed{A\mathbf{q} = \mathbf{p}}$$



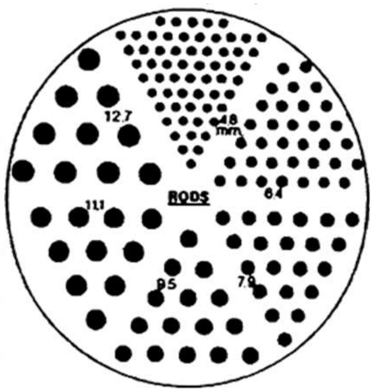
1979
Nobel
Prize in
Medicine

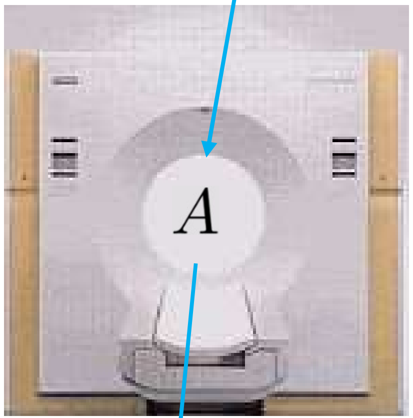
https://www.imaginis.com/ct-scan/how-does-ct-work

Jasczcak Phantom

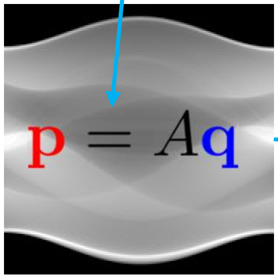
https://www.nuclear-shields.com/

$\mathbf{q}$





A



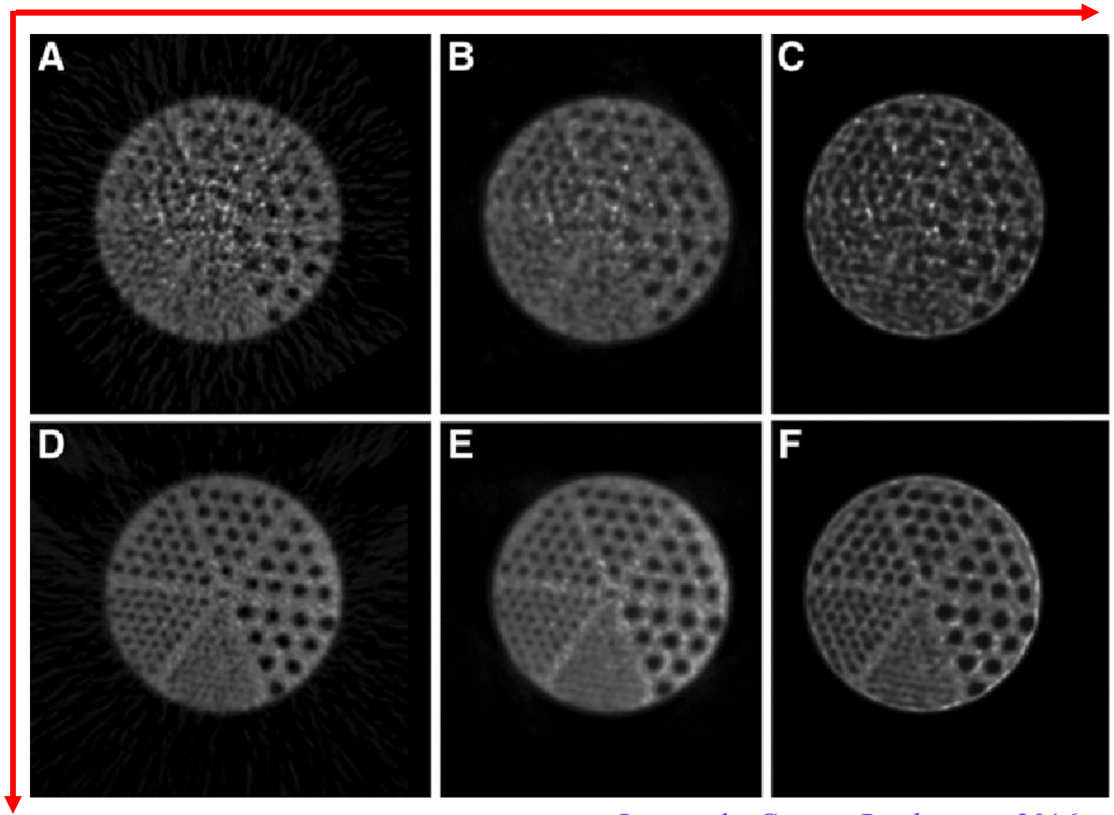
$\mathbf{p} = A\mathbf{q}$

$A^{-1}\mathbf{p}$

Toward better $\mathbf{q}$

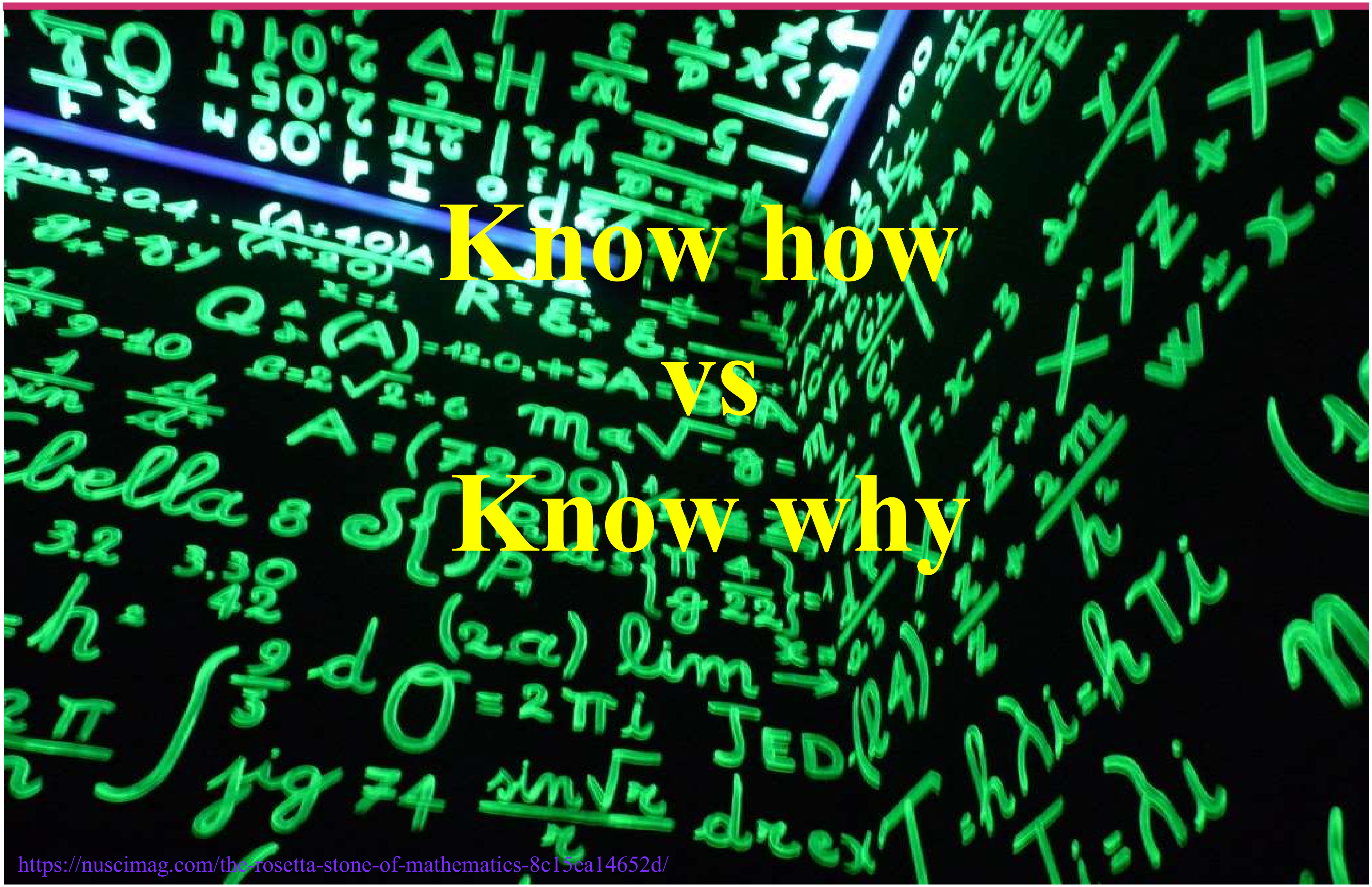
better software (math algo)

better hardware



Lee et al., *Cancer Biotherapy*, 2016

Why need Math?



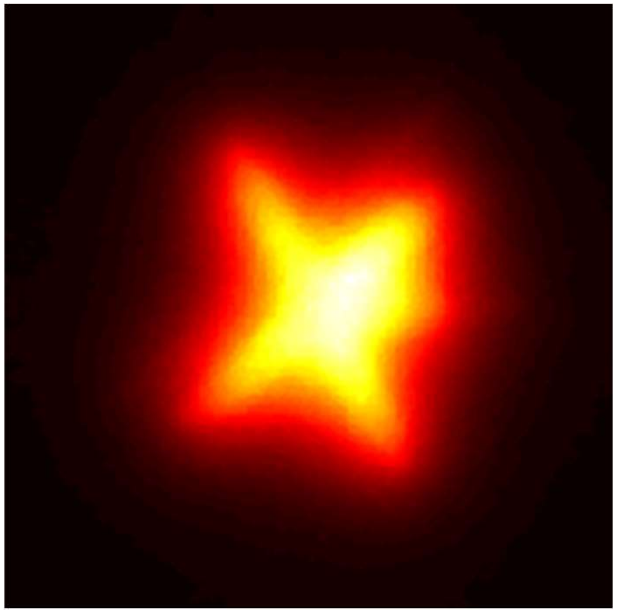
Know how

vs

Know why

https://nuscimag.com/the-rosetta-stone-of-mathematics-8c15ea14652d/

Ground-based Astronomy Revisit



Satellite from
Ground-based telescope

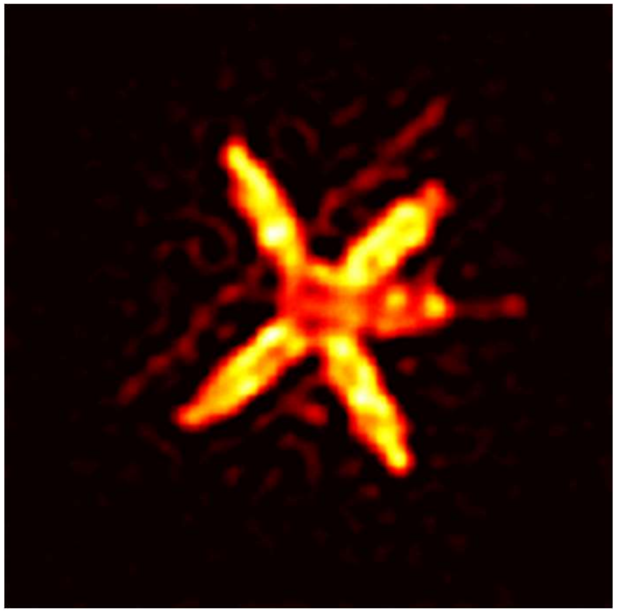


Image after
reconstruction
(1993)

Can we do better?

Multi-frame Super-resolution

A 352-by-288 video
from a video recorder

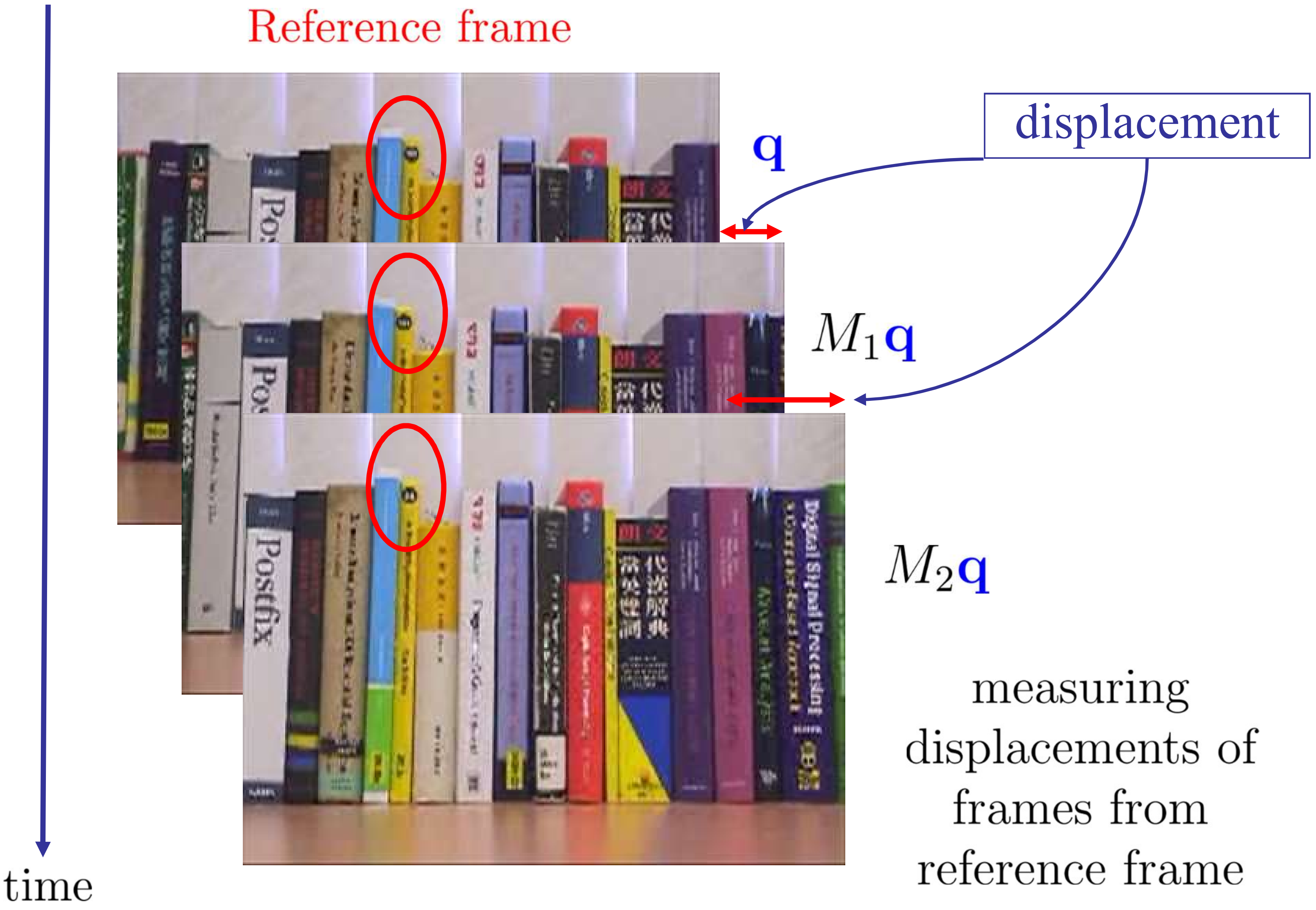
30 frames/second

Bilinear interpolation
using 21 frames

Eighth frame
from 21 frames

Chan, Shen, and Xia, *Applied Computational Harmonic Analysis*, 2007

Aligning the Frames



Reference frame

displacement

$\mathbf{q}$

$M_1\mathbf{q}$

$M_2\mathbf{q}$

time

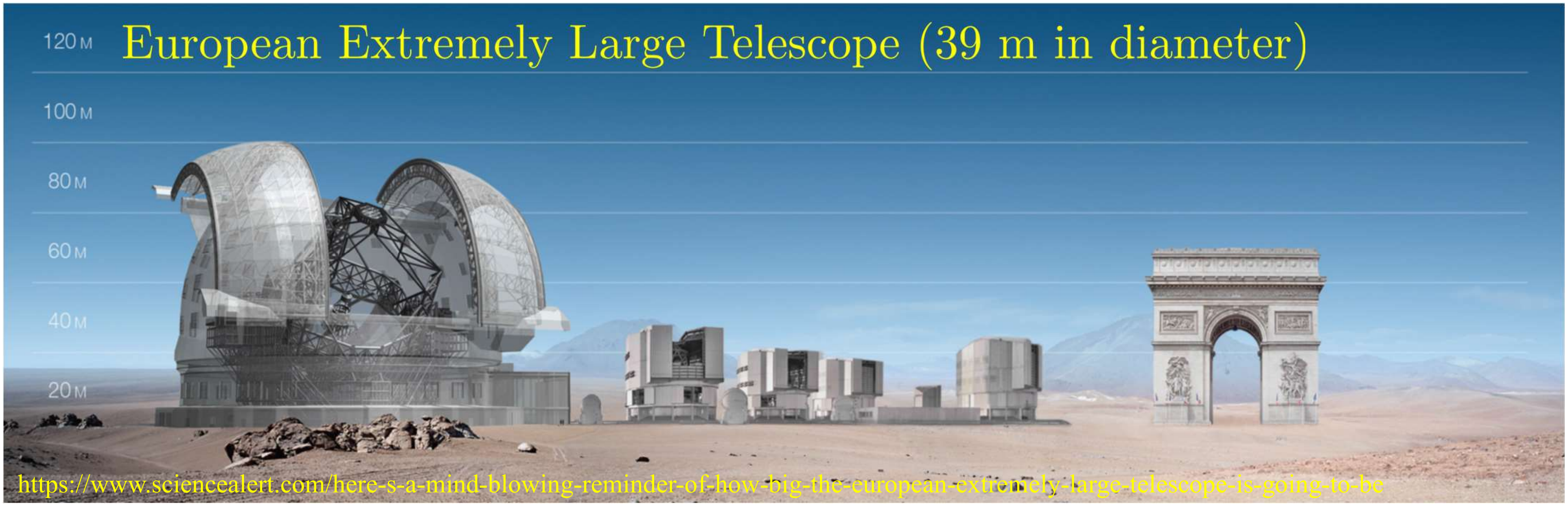
measuring
displacements of
frames from
reference frame

Super-resolution for Ground-based Astronomy

How to get multiple frames of space objects?

☐ object itself is moving!

☐ move the telescope

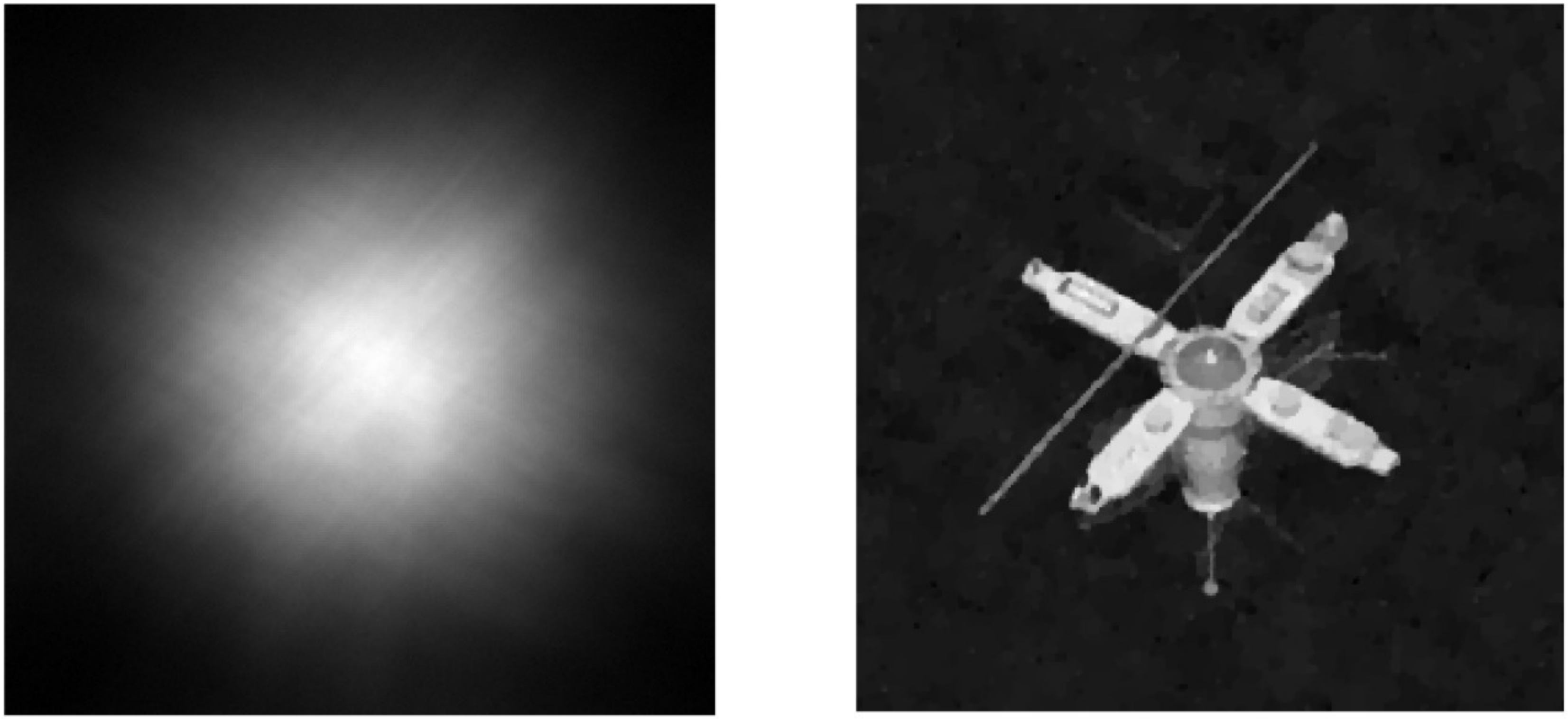

European Extremely Large Telescope (39 m in diameter)

https://www.sciencealert.com/here-s-a-mind-blowing-reminder-of-how-big-the-european-extremely-large-telescope-is-going-to-be

☐ air is moving!

Super-resolution of Satellite

Clear image obtained by moving the telescope:



Observed image
by Earth telescope

Our Method

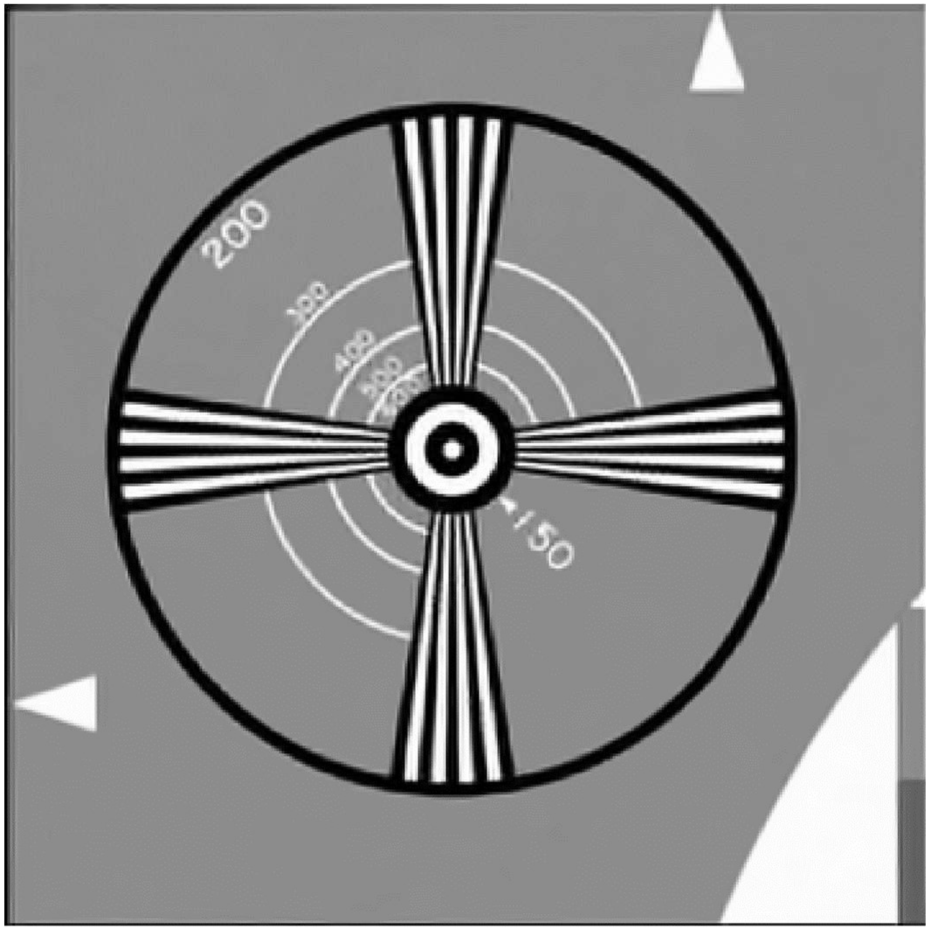
Chan, Yuan & Zhang, *J. Opt. Soc. Am. A, 29 (2012).*

Ke, Wagner, Ramlau, & Chan, *SIAM Journal on Imaging Science (2020).*

Video Still Enhancement

19 frames
of size
57 × 49

Low
resolution
video



Low resolution zoomed-in image

Our method

Video source: https://users.soe.ucsc.edu/~milanfar/software/sr-datasets.html

Low resolution zoom-in image

21 frames of size 96 × 128

Low resolution video

Our method

Input video

Qian *et al.* Siggraph 09

Upsampled by bicubic

Our method

Artificial Intelligence or Human Intelligence

Let us solve the quadratic equation:

$$6x^2 - 78x + 252 = 0$$

Solution is

$$x = \frac{78 \pm \sqrt{78^2 - 4 \cdot 6 \cdot 252}}{2 \cdot 6} = 6,7$$

Who solved it?

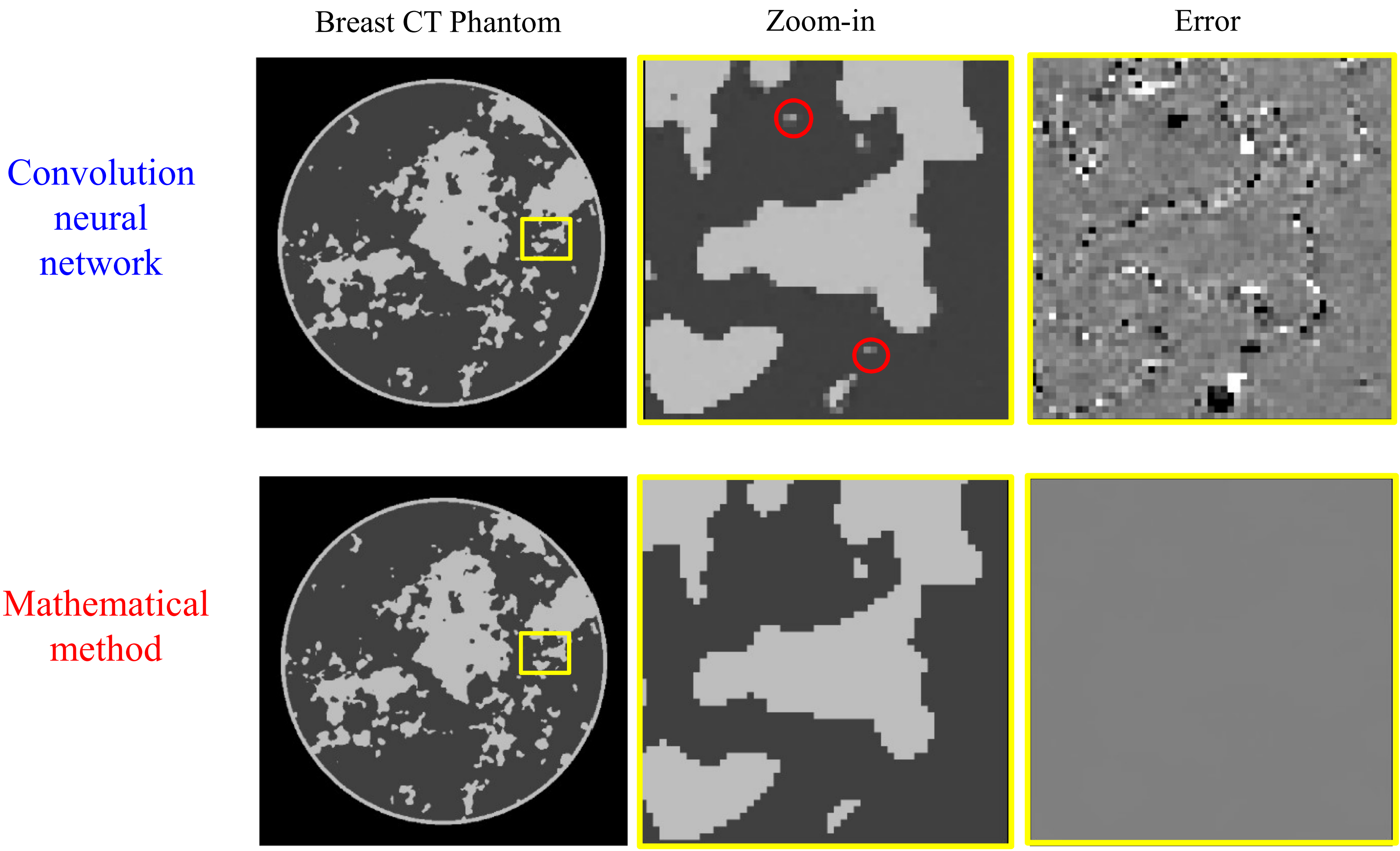
Computer

or

Math

AI algorithms are invented by homosapiens!

Plug-and-Play Deep Neural Network



Do CNNs solve the CT inverse problem? Sidky, Lorente, Brankov, & Pan, ArXiv, 2020.

Plug-and-Play or Plug-and-Pray ?

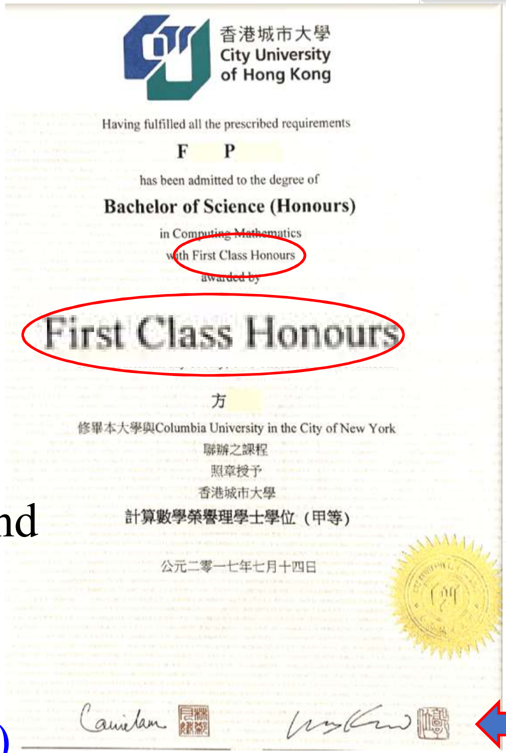
New Programmes

- JUPAS option (JS1200) — Global Research Enrichment and Technopreneurship (GREAT)
- Double-degree Joint Bachelor's Programmes — with Columbia University, University of Edinburgh, and University of Manchester
- Double-degree Bachelor's Programme — Law and Math (confirm by Nov, 20)

STEM Initiatives

- CityU Science Challenges – mock HKDSE assessments (Jan, 2021)
- Popular Stem Talks by our Professors

https://www.cityu.edu.hk/csci/new-programmes-and-stem-initiatives



香港城市大學
City University
of Hong Kong

Having fulfilled all the prescribed requirements

F [redacted] P [redacted]

has been admitted to the degree of

Bachelor of Science (Honours)

in Computing Mathematics

with First Class Honours

awarded by

First Class Honours

方 [redacted]

修畢本大學與Columbia University in the City of New York

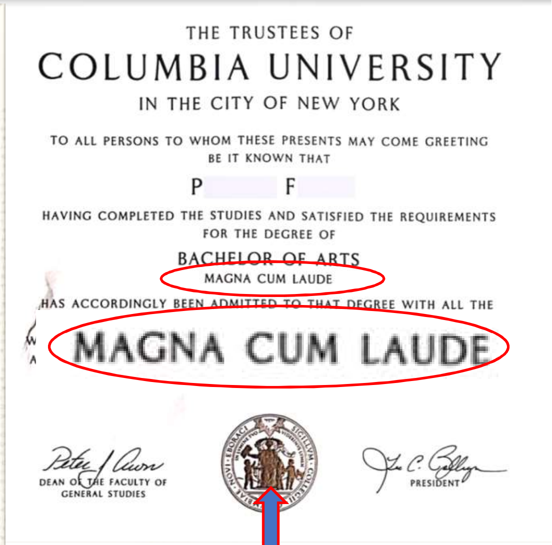
聯辦之課程

照章授予

香港城市大學

計算數學榮譽理學士學位（甲等）

公元二零一七年七月十四日



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IN THE CITY OF NEW YORK

TO ALL PERSONS TO WHOM THESE PRESENTS MAY COME GREETING

BE IT KNOWN THAT

P [redacted] F [redacted]

HAVING COMPLETED THE STUDIES AND SATISFIED THE REQUIREMENTS

FOR THE DEGREE OF

BACHELOR OF ARTS

MAGNA CUM LAUDE

HAS ACCORDINGLY BEEN ADMITTED TO THAT DEGREE WITH ALL THE

MAGNA CUM LAUDE

DEAN OF THE FACULTY OF GENERAL STUDIES

PRESIDENT

Years 3 & 4 @ partner U

Years 1 & 2 @ CityU

Graduation Ceremony @ Columbia U (2019)

Thank You!

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